The Algebra of Expectations

In my paper on random variables I gave examples of random variables created from other random variables. In this paper I prove formulas for computing the expectation and variance of random variables created from other random variables.

Theorem 1

$E(X + a) = E(X) + a$, where a is a constant.

Proof:

$$\begin{aligned} E(X + a) &= (x_1 + a)p(x_1) + (x_2 + a)p(x_2) + \ldots + (x_k + a)p(x_k) \\ &= x_1p(x_1) + x_2p(x_2) + \ldots + x_kp(x_k) + a(p(x_1) + p(x_2) + \ldots + p(x_k)) \\ &= E(X) + a \end{aligned}$$

□

Theorem 2

$E(aX) = aE(X)$, where a is a constant

Proof:

$$\begin{aligned} E(aX) &= ax_1p(x_1) + ax_2p(x_2) + \ldots + ax_kp(x_k) \\ &= a(x_1p(x_1) + x_2p(x_2) + \ldots + x_kp(x_k)) \\ &= aE(X) \end{aligned}$$

□

Theorem 3

$E(X_1+X_2+ \ldots +X_N) = E(X_1) + E(X_2) + \ldots + E(X_N)$

Proof:

$$E(X_1+X_2+ \ldots +X_N) = \sum[X_1(\bullet)+X_2(\bullet)+ \ldots +X_N(\bullet)]p(\bullet)$$

$$= \sum[X_1(\bullet)p(\bullet)+X_2(\bullet)p(\bullet)+ \ldots +X_N(\bullet)p(\bullet)]$$

$$= \sum X_1(\bullet)p(\bullet) + \sum X_2(\bullet)p(\bullet) + \ldots + \sum X_N(\bullet)p(\bullet)$$

$$= E(X_1) + E(X_2) + \ldots + E(X_N)$$

□

Comment

Now you know why I introduced an alternative definition of E(X) in my paper about random variables. The usual way to prove this would be to prove it for $X_1 +X_2$ using a complicated summation and use mathematical induction to prove it for $X_1 +X_2 + \ldots +X_N$. This way, I can prove it all at once without using mathematical induction.

Theorem 4

If X and Y are independent random variables then

$E(XY) = E(X)E(Y)$

Proof:

let k be the number of values that X can take on and let m be the number of values that Y can take on.
Since X and Y are independent we have $p(x_i y_j) = p(x_i)p(y_j)$.
So running through all the combinations of $x_i y_j$' s we get:

$$
\begin{aligned}
E(XY) = {} & x_1y_1p(x_1)p(y_1) + x_1y_2p(x_1)p(y_2) + \ldots + x_1y_mp(x_1)p(y_m) \\
& + x_2y_1p(x_2)p(y_1) + x_2y_2p(x_2)p(y_2) + \ldots + x_2y_mp(x_2)p(y_m) \\
& + x_3y_1p(x_3)p(y_1) + x_3y_2p(x_3)p(y_2) + \ldots + x_3y_mp(x_3)p(y_m) \\
& + \quad * \qquad\qquad * \qquad\qquad * \\
& \quad\;\; * \qquad\qquad * \qquad\qquad * \\
& \quad\;\; * \qquad\qquad * \qquad\qquad * \\
& \underline{+x_ky_1p(x_k)p(y_1) + x_ky_2p(x_k)p(y_2) + \ldots + x_ky_mp(x_k)p(y_m)} \\
& \quad E(X)y_1p(y_1) + E(X)y_2p(y_2) + \ldots + E(X)y_mp(y_m)
\end{aligned}
$$

Adding the column totals we get $E(X)E(Y)$

So $E(XY) = E(X)E(Y)$

□

Theorem 5

$V(aX) = a^2 V(X)$ Where a is a constant

Proof:

$$V(aX) = \sum_{i=1}^{k} (ax_i - aE(X))^2 p(x_i)$$

$$= \sum_{i=1}^{k} a^2(x_i - E(X))^2 p(x_i)$$

$$= a^2 \sum_{i=1}^{k} (x_i - E(X))^2 p(x_i)$$

$$= a^2 V(X)$$

□

Theorem 6

If X and Y are independent

$E[(X-E(X))(Y-E(Y))] = 0$

Proof:

$$E[(X-E(X))(Y-E(Y))] = E[XY - E(X)Y - E(Y)X + E(X)E(Y)]$$
$$= E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y)$$
$$= E(XY) - E(X)E(Y)$$

So since X and Y are independent, then $E(XY) = E(X)E(Y)$ and $E[(X-E(X))(Y-E(Y))] = 0$.

□

Theorem 7

$$V(X_1+X_2+ \dots +X_N) = \sum_{i=1}^{N} V(X_i) \quad +2\sum_{i<j} E[(X_i-E(X_i))(X_j-E(X_j))]$$

Proof:

$$V(X_1+X_2+ \dots +X_N) = E[X_1+X_2+ \dots +X_N - E(X_1+X_2+ \dots +X_N)]^2$$
$$= E[(X_1 -E(X_1)) +(X_2 -E(X_2)) + \dots + (X_N -E(X_N))]^2$$
$$=E[(X_1 -E(X_1))^2 +(X_2 -E(X_2))^2 + \dots +(X_N -E(X_N))^2 +2\sum_{i<j} ((X_i-E(X_i))(X_j-E(X_j))]$$
$$=E[X_1 -E(X_1)]^2 +E[X_2 -E(X_2)]^2 + \dots +E[X_N -E(X_N)]^2 +2\sum_{i<j} E[(X_i-E(X_i))(X_j-E(X_j))]$$
$$= \sum_{i=1}^{N} V(X_i) \quad +2\sum_{i<j} E[(X_i-E(X_i))(X_j-E(X_j)]$$

□

Theorem 8

If X_1 , X_2 , ... , X_N are pairwise independent, then

$$V(X_1+X_2+ \dots +X_N) = \sum_{i=1}^{N} V(X_i)$$

Proof:

This follows from Theorem 7 and Theorem 6.

□

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